Moving Fronts in Integro-Parabolic Reaction-Diffusion-Advection Equations

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Abstract. The Neumann boundary value problem for a class of singularly perturbed integro-parabolic equations is considered. An asymptotic expansion of a new class of solutions of moving front type is constructed, and a theorem of existence of such solutions is proved.

1. Introduction

Mathematical problems concerning reaction-advection-diffusion equations describe many important practical applications in chemical kinetics, synergetics, astrophysics, biology, etc. In many important cases the solutions of these problems feature internal layers (see [1] and references therein). Recently there is an increasing interest to more complicated models, which include the effects of feedback or non-local interaction. These models are represented by integro-differential equations (see [2], [3], [4]).

In this work we consider the following family of ε-depending initial boundary value problems

$$L[u] \equiv -\varepsilon\frac{\partial u}{\partial t}(x,t,\varepsilon) + \varepsilon^2\frac{\partial^2 u}{\partial x^2}(x,t,\varepsilon) - \varepsilon A(x,\varepsilon)\frac{\partial u}{\partial x}(x,t,\varepsilon) - \\ - \int_a^b g(u(x,t,\varepsilon), u(s,t,\varepsilon), x, s)\, ds = 0, \quad a < x < b, \tag{1}$$

$$\frac{\partial u}{\partial x}(a,t,\varepsilon) = 0, \quad \frac{\partial u}{\partial x}(b,t,\varepsilon) = 0, \quad u(x,0,\varepsilon) = u^0(x,\varepsilon) \tag{2}$$

and investigate the existence of moving internal layer solutions (fronts). Here, $A(x,\varepsilon)$ and $g(u,v,x,s,\varepsilon)$ are sufficiently smooth functions (their actual degree of smoothness is specified below), $u^0(x,\varepsilon)$ is some initial function of front type, and $\varepsilon > 0$ is a small parameter.

The corresponding stationary boundary value problem for the case $A \equiv 0$ was considered in [5]. Our results develop and extend methods proposed in [5] and [6] to a new more complicated class of problems. This work can be also considered as an extension of results of [7] to nonlocal BVP's. Some related problems where travelling wave solutions for nonlocal problem with bistable nonlinearity and linear integral term was considered in [4](see also references therein).

The reduced equation

$$\int\limits_a^b g(u(x,t), u(s,t), x, s)\, ds = 0 \tag{3}$$

is an integral equation. Following to the investigation in [5], in order to have a solution to (1), (2) of internal layer type we suppose to have a family of discontinuous solutions of problem (3). Namely, we assume

Condition I. *There exist two functions*

$$\begin{aligned} &\varphi^{(-)} \in C(\Omega^{(-)}), \quad \textit{where} \quad \Omega^{(-)} \equiv \{(x,y) : a \le x \le y \le b\}, \\ &\varphi^{(+)} \in C(\Omega^{(+)}), \quad \textit{where} \quad \Omega^{(+)} \equiv \{(x,y) : a \le y \le x \le b\}, \end{aligned}$$

which for every $y \in (a,b)$ satisfy $\varphi^{(-)}(y,y) < \varphi^{(+)}(y,y)$ and the system of the two coupled integral equations

$$\begin{aligned} &\int\limits_a^y g(\varphi^{(-)}(x,y), \varphi^{(-)}(s,y), x, s, 0)\, ds + \\ &\qquad + \int\limits_y^b g(\varphi^{(-)}(x,y), \varphi^{(+)}(s,y), x, s, 0)\, ds = 0, \quad a < x < y, \\ &\int\limits_a^y g(\varphi^{(+)}(x,y), \varphi^{(-)}(s,y), x, s, 0)\, ds + \\ &\qquad + \int\limits_y^b g(\varphi^{(+)}(x,y), \varphi^{(+)}(s,y), x, s, 0)\, ds = 0, \quad y < x < b. \end{aligned}$$

We also assume that $\int_a^b g_u(\varphi^{(i)}(x,y), \varphi(s,y), x, s, \varepsilon) ds > 0$ for all $x, y \in [a,b]$ and $i = -, +$, where

$$\varphi(x,y) \equiv \begin{cases} \varphi^{(-)}(x,y) & \textit{for} \quad x \in \Omega^{(-)}, \\ \varphi^{(+)}(x,y) & \textit{for} \quad x \in \Omega^{(+)}, \end{cases}$$

and that there exist a continuous function $\varphi_0(x)$ such that for all $x \in [a,b]$

$$\varphi^{(-)}(x,x) < \varphi_0(x) < \varphi^{(+)}(x,x) \ \textit{and} \int\limits_a^b g_u(\varphi_0(x), \varphi_0(s), x, s, \varepsilon) ds < 0$$

(g_u is derivative of g with respect to the first argument).

The goal of our paper is to show that, under some additional assumptions, the solution to problem (1), (2) is of moving internal layer type

$$u(x,t,\varepsilon) \approx \varphi(x, x^*(t,\varepsilon)) \text{ for } \varepsilon \approx 0,$$

where the location $x^*(t,\varepsilon)$ of the internal layer is the point of intersection of the solution u with the function φ_0:

$$u(x^\star(t,\varepsilon), t, \varepsilon) = \varphi_0(x^\star(t,\varepsilon)). \tag{4}$$

In what follows the functions u and x^* are the unknowns to be determined as solutions to (1), (2), (4).

2. The formal asymptotic expansion of internal layer solutions

In this section we construct asymptotic expansions of internal layer solutions to problem (1), (2). In order to approximate the introduced location of the front $x^*(t,\varepsilon)$ we use the following ansatz

$$X^*(t,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k x_k(t).$$

We also assume that the initial function $u^0(x,\varepsilon)$ is of front type, i.e. such that for small ε $u^0(x,\varepsilon)$ is close to $\varphi(x,x_{00})$ for $a\le x<x_{00}$ and $x_{00}<x\le b$, where x_{00} is some point from (a,b) where initially the front is located, that is we assume

$$X^*(0,\varepsilon)=x_{00}.$$

In order to construct the formal asymptotic expansion we rewrite the initial boundary value problem (1),(2),(4) as a system of two inital boundary value problems:

$$-\varepsilon\frac{\partial u}{\partial t}+\varepsilon^2\frac{\partial^2 u}{\partial x^2}-\varepsilon A(x,\varepsilon)\frac{\partial u}{\partial x}-\int_a^b g(u(x,t),u(s,t.\varepsilon),x,s)\,ds=0,\quad a<x<X^*(t,\varepsilon),\tag{5}$$

$$\frac{\partial u}{\partial x}(a,t)=0,\ u(X^*(t,\varepsilon),t,\varepsilon)=\varphi_0(X^*(t,\varepsilon)),\ u(x,0)=u^0(x,\varepsilon),\ a\le x\le X^*(t,\varepsilon),\tag{6}$$

$$-\varepsilon\frac{\partial u}{\partial t}+\varepsilon^2\frac{\partial^2 u}{\partial x^2}-\varepsilon A(x,\varepsilon)\frac{\partial u}{\partial x}-\int_a^b g(u(x,t),u(s,t),x,s)\,ds=0,\quad X^*(t,\varepsilon)<x<b,\tag{7}$$

$$u(X^*(t,\varepsilon),t,\varepsilon)=\varphi_0(X^*(t,\varepsilon)),\quad \frac{\partial u}{\partial x}(b,t)=0,\ u(x,0)=u^0(x,\varepsilon),\ X^*(t,\varepsilon)\le x\le b.\tag{8}$$

We construct boundary layer asymptotic expansions for the solutions to the two initial boundary value problems (5),(6)and (7),(8) above and use the boundary layers near the point $X^*(t,\varepsilon)$ to describe the internal layer solution of the original problem (1),(2). In the principal part of construction we follow the paper [5], where the reader can find some additional details of the developing of our representations which are omitted here for shortness.

To build the formal asymptotics $U(x,t,\varepsilon)$ of the seeking solution u we shall use the following ansatz

$$U(x,t,\varepsilon)=\begin{cases}U^{(-)}(x,t,\varepsilon)=\overline{u}^{(-)}(x,X^*(t,\varepsilon),t,\varepsilon)+Q^{(-)}(\xi,t,\varepsilon)+\Pi^{(-)}(\xi^{(-)},\varepsilon)\\ \text{for}\quad a\le x\le X^*(t,\varepsilon),\\ U^{(+)}(x,t,\varepsilon)=\overline{u}^{(+)}(x,X^*(t,\varepsilon),t,\varepsilon)+Q^{(+)}(\xi,t,\varepsilon)+\Pi^{(+)}(\xi^{(+)},\varepsilon)\\ \text{for}\quad X^*(t,\varepsilon)\le x\le b.\end{cases}\tag{9}$$

Here $\overline{u}^{(\pm)}(x,y,\varepsilon)$ is the regular part, $Q^{(\pm)}(\xi,t,\varepsilon)$ are the internal layer parts and $\Pi^{(\pm)}(\xi^{(\pm)},\varepsilon)$ are the boundary parts. The parts $Q^{(\pm)}(\xi,t,\varepsilon)$ serve to describe the quick transition layer in a small vicinity of the point $X(t,\varepsilon) \in (a,b)$ and thus depend on the stretched variable

$$\xi = \frac{x - X^*(t,\varepsilon)}{\varepsilon}.$$

The functions $\Pi^{(\pm)}(\xi^{(\pm)},\varepsilon)$ describe the boundary layers on the left and right sides of interval (a,b) and depend on the stretched variables

$$\xi^{(-)} = \frac{x-a}{\varepsilon} \text{ and } \xi^{(+)} = \frac{b-x}{\varepsilon}$$

accordingly. Each term of the presented ansatz is treated as an integer power series with respect to the small parameter ε, namely

$$\overline{u}^{(\pm)}(x,y,t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \overline{u}_k^{(\pm)}(x,y,t), \quad Q^{(\pm)}(\xi,t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k Q_k^{(\pm)}(\xi,t),$$

$$\Pi^{(\pm)}(\xi^{(\pm)},\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \Pi_k^{(\pm)}(\xi^{(\pm)}).$$

The construction of the functions $\Pi_k^{(\pm)}$ is described in details in [8]. These functions are not depending on variable t, and therefore they do not have influence on the moving internal layer. We note that in the case of Neumann boundary conditions we would have $\Pi_0^{(\pm)} \equiv 0$.

Remark 1. *It will be shown below that the functions $Q_k^{(\pm)}(\xi,t)$ depend of t just throw dependence of $X^*(t,\varepsilon)$. Therefore for $U(x,t,\varepsilon)$ and $Q^{(\pm)}(\xi,t,\varepsilon)$ we also use the notations $U(x,X^*(t,\varepsilon),\varepsilon)$ and $Q^{(\pm)}(\xi,X^*(t,\varepsilon),\varepsilon)$.*

In order to find the terms of the asymptotic expansion of X^* we use the condition of C^1-matching of the asymptotics at the point $X^*(t,\varepsilon)$:

$$U^{(-)}(X^*(t,\varepsilon),t,\varepsilon) = U^{(+)}(X^*(t,\varepsilon),\varepsilon), \qquad \varepsilon\frac{\partial U^{(-)}}{\partial x}(X^*(t,\varepsilon),t,\varepsilon) = \varepsilon\frac{\partial U^{(+)}}{\partial x}(X^*(t,\varepsilon),t,\varepsilon).$$

This is equivalent to

$$\begin{aligned} &Q_0^{(-)}(0,t) + \overline{u}_0^{(-)}(x_0(t),x_0(t)) + \\ &\quad + \sum_{k=1}^{\infty} \varepsilon^k \left(Q_k^{(-)}(0,t) + \overline{u}_k^{(-)}(x_0(t),x_0(t)) + x_k(t)w^{(-)}(t) + M_k^{(-)}(t) \right) = \\ &\quad = Q_0^{(+)}(0,t) + \overline{u}_0^{(+)}(x_0(t),x_0(t)) + \\ &\quad + \sum_{k=1}^{\infty} \varepsilon^k \left(Q_k^{(+)}(0,t) + \overline{u}_k^{(+)}(x_0(t),x_0(t)) + x_k(t)w^{(+)}(t) + M_k^{(+)}(t) \right) \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\frac{\partial Q_0^{(-)}}{\partial \xi}(0,t) + \sum_{k=1}^{\infty} \varepsilon^k \left(\frac{\partial Q_k^{(-)}}{\partial \xi}(0,t) + N_k^{(-)}(t) \right) = \\ &\quad = \frac{\partial Q_0^{(+)}}{\partial \xi}(0,t) + \sum_{k=1}^{\infty} \varepsilon^k \left(\frac{\partial Q_k^{(+)}}{\partial \xi}(0,t) + N_k^{(+)}(t) \right), \end{aligned} \tag{11}$$

where

$$w^{(\pm)}(t) \equiv \frac{\partial \overline{u}_0^{(\pm)}}{\partial x}(x_0(t), x_0(t)) + \frac{\partial \overline{u}_0^{(\pm)}}{\partial y}(x_0(t), x_0(t)).$$

Here $M_k^{(\pm)}$ and $N_k^{(\pm)}$ are certain functions recurrently expressed in terms of the preceding orders of the asymptotics, in particular

$$M_1^{(\pm)}(t) = 0, \qquad N_1^{(\pm)}(t) = \frac{\partial \overline{u}_0^{(\pm)}}{\partial x}(x_0(t), x_0(t)).$$

To formulate problems that determine the terms appearing in this series, it is necessary to represent the equations in the coupled system of integro-differential equations in the form of a sum of regular and boundary layer parts. For simplicity we shall omit indices $(-)$ and $(+)$ in what follows if it is possible without misunderstanding. We represent integral term as the sum

$$\int_a^b g(U(x,t,\varepsilon), U(s,t,\varepsilon), x, s, \varepsilon)\, ds = \sum_{k=1}^{2} \overline{L}_k(x, X^*(t,\varepsilon), \varepsilon) + \sum_{k=1}^{2} QL_k(\xi, t, \varepsilon). \tag{12}$$

Here and in what follows, the following notation is used:

$$\begin{aligned}
\overline{L}_1(x,y,\varepsilon) \;&\equiv\; \int_a^b g(\overline{u}(x,y,\varepsilon), \overline{u}(s,y,\varepsilon), x, s, \varepsilon)\, ds, \\
\overline{L}_2(x,y,\varepsilon) \;&\equiv\; \int_a^b \left(g(\overline{u}(x,y,\varepsilon), U(s,y,\varepsilon), x, s, \varepsilon) - g(\overline{u}(x,y,\varepsilon), \overline{u}(s,y,\varepsilon), x, s, \varepsilon)\right)\, ds, \\
QL_1(\xi,t,\varepsilon) \;&\equiv\; \int_a^b \left(g(U(x,t,\varepsilon), \overline{u}(s, X^*(t,\varepsilon), \varepsilon), x, s, \varepsilon)\right. \\
&-\; \left. g(\overline{u}(x, X^*(t,\varepsilon), \varepsilon), \overline{u}(s, X^*(t,\varepsilon), \varepsilon), x, s, \varepsilon)\right)\, ds
\end{aligned}$$

and

$$\begin{aligned}
QL_2(\xi,t,\varepsilon) \equiv \int_a^b &\left(g(U(x,t,\varepsilon), U(s,t,\varepsilon), x, s, \varepsilon) - g(\overline{u}(x, X^*(t,\varepsilon), \varepsilon), U(s,t,\varepsilon), x, s, \varepsilon) - \right. \\
&- \left. g(u(x,t,\varepsilon), \overline{u}(s, X^*(t,\varepsilon), \varepsilon), x, s, \varepsilon) + g(\overline{u}(x, X^*(t,\varepsilon), \varepsilon), \overline{u}(s, x^\star(t,\varepsilon), \varepsilon), x, s, \varepsilon)\right)\, ds.
\end{aligned}$$

Let us separately transform each of the above terms. Represent the first term $\overline{L}_1$ in the form

$$\begin{aligned}
\overline{L}_1(x,y,\varepsilon) = &\int_a^b g(\overline{u}_0(x,y), \overline{u}_0(s,y), x, s, 0)\, ds + \\
&+ \sum_{k=1}^{\infty} \varepsilon^k \left(P(x,y)\overline{u}_k + \int_a^b g_v(\overline{u}_0(x,y), \overline{u}_0(s,y), x, s, 0)\overline{u}_k(s,y)\, ds + D_k(x,y) \right),
\end{aligned}$$

where

$$P(x,y) \equiv \int\limits_a^b g_u(\overline{u}_0(x,y), \overline{u}_0(s,y), x, s, 0)\, ds,$$

(g_v is derivative of g for second argument), and $D_k(x,y)$ are recurrently expressed in terms of the preceding orders of the asymptotics, in particular,

$$D_1(x,y) = \int\limits_a^b g_\varepsilon(\overline{u}_0(x,y), \overline{u}_0(s,y), x, s, 0)\, ds.$$

The term $\overline{L}_2$ is transformed in a somewhat different way. Before expressing it as a series in the small parameter ε, we change the integration variable by the formula $s = X^*(t,\varepsilon) + \varepsilon\tau$ (τ is the stretched variable). As a result, we obtain

$$\overline{L}_2(x, X^*(t,\varepsilon), \varepsilon) =$$

$$\varepsilon \int\limits_{(a-X^*(t,\varepsilon))/\varepsilon}^{(b-X^*(t,\varepsilon))/\varepsilon} [g(\overline{u}(x, X^*(t,\varepsilon), \varepsilon), \overline{u}(X^*(t,\varepsilon) + \varepsilon\tau, X^*(\varepsilon), \varepsilon) + Q(\tau, t, \varepsilon), x, X^*((t,\varepsilon) + \varepsilon\tau, \varepsilon)$$

$$-g(\overline{u}(x, X^*(t,\varepsilon), \varepsilon), \overline{u}(X^*((t,\varepsilon) + \varepsilon\tau, X^*((t,\varepsilon), \varepsilon), x, X^*((t,\varepsilon) + \varepsilon\tau, \varepsilon)]\ d\tau$$

$$\equiv \varepsilon \int\limits_{(a-X^*(t,\varepsilon))/\varepsilon}^{(b-X^*(t,\varepsilon))/\varepsilon} [\dots]\ d\tau = \varepsilon \int\limits_{-\infty}^{+\infty} [\dots]\ d\tau - \varepsilon \int\limits_{-\infty}^{(a-X^*((t,\varepsilon))/\varepsilon} [\dots]\ d\tau - \varepsilon \int\limits_{(b-X^*((t,\varepsilon))/\varepsilon}^{+\infty} [\dots]\ d\tau =$$

$$= \varepsilon \int\limits_{-\infty}^{+\infty} [g(\overline{u}_0(x, X^*(t,\varepsilon)), \overline{u}_0(x_0, x_0) + Q_0(\tau, t), x, x_0, 0) -$$

$$g(\overline{u}_0(x, X^*(t,\varepsilon)), \overline{u}_0(x_0, x_0), x, x_0, 0)]\ d\tau +$$

$$+ \sum_{k=1}^{\infty} \varepsilon^{k+1} T_{k+1}(x, X^*((t,\varepsilon)) - \Phi_1(x, X^*((t,\varepsilon), \varepsilon) - \Phi_2(x, X^*((t,\varepsilon), \varepsilon).$$

Here, $T_k(x, X^*((t,\varepsilon))$ are certain functions recurrently expressed in terms of the preceding orders of the asymptotics and

$$\Phi_1(x, X^*((t,\varepsilon), \varepsilon) \equiv \varepsilon \int\limits_{-\infty}^{(a-X^*((t,\varepsilon))/\varepsilon} [\dots]\ d\tau, \quad \Phi_2(x, X^*(t,\varepsilon), \varepsilon) \equiv \varepsilon \int\limits_{(b-X^*((t,\varepsilon))/\varepsilon}^{+\infty} [\dots]\ d\tau.$$

It is shown below that all the boundary functions $Q_k(\xi, t)$ have an exponential estimate at infinity. Under this condition, the integrand in the definition of $\Phi_1(x, X^*(t,\varepsilon), \varepsilon)$ also satisfies this estimate. Then, the estimate $|\Phi_1(x, X^*(t,\varepsilon), \varepsilon)| \le (\varepsilon C/\nu) e^{\nu(a-X^*((t,\varepsilon))/\varepsilon}$ is valid, therefore, the relationship $\Phi_1(x, X^*(t,\varepsilon), \varepsilon) = o(\varepsilon^n)$ is fulfilled for all $n \ge 0$ as $\varepsilon \to +0$. Analogously, $\Phi_2(x, X^*(t,\varepsilon), \varepsilon) = o(\varepsilon^n)$ for all $n \ge 0$ as $\varepsilon \to +0$. Thus it is possible to neglect terms $\Phi_1(x, X^*(t,\varepsilon), \varepsilon)$ and $\Phi_2(x, X^*(t,\varepsilon), \varepsilon)$ in comparison with any other term with a power dependence on ε.

Using the conventional scheme after transformation of QL_1, we obtain

$$QL_1(\xi, t, \varepsilon) = \int\limits_a^b (g(\overline{u}_0(x_0, x_0) + Q_0(\xi, t), \overline{u}_0(s, x_0), x_0, s, 0) -$$
$$-g(\overline{u}_0(x_0, x_0), \overline{u}_0(s, x_0), x_0, s, 0))\ ds + \sum_{k=1}^{\infty} \varepsilon^k S_k(\xi, t).$$

Finally, when transforming the term QL_2, the same sequence of operations as in the case with the term $\overline{L}_2$ is required. As a result, we obtain

$$QL_2(\xi, t, \varepsilon) = \varepsilon \int\limits_{-\infty}^{+\infty} (g(\overline{u}_0(x_0, x_0) + Q_0(\xi, t), \overline{u}_0(x_0, x_0) + Q_0(\tau, t), x_0, x_0, 0) -$$
$$-g(\overline{u}_0(x_0, x_0), \overline{u}_0(x_0, x_0) + Q_0(\tau, t), x_0, x_0, 0) -$$
$$-g(\overline{u}_0(x_0, x_0) + Q_0(\xi, t), \overline{u}_0(x_0, x_0), x_0, x_0, 0) +$$
$$+g(\overline{u}_0(x_0, x_0), \overline{u}_0(x_0, x_0), x_0, x_0, 0))\ d\tau + \sum_{k=1}^{\infty} \varepsilon^{k+1} \Theta_k(\xi, t) - \Psi(\xi, t, \varepsilon). \qquad (13)$$

Here $\Theta_k(\xi, t)$ are certain functions recurrently expressed in terms of the preceding orders of the asymptotics and the function $\Psi(\xi, t, \varepsilon)$ has the same origin as functions $\Phi_1(x, y, \varepsilon)$ and $\Phi_2(x, y, \varepsilon)$ in the case of the expansion for $\overline{L}_2$. By analogy with the above discussion, it can be shown that, in the subsequent reasoning, $\Psi(\xi, t, \varepsilon) = o(\varepsilon^n)$ for all $n \geq 0$, therefore, the function Ψ can be neglected in comparison with any term with a power dependence on ε.

Applying the differential operator $D \equiv \varepsilon^2\ \frac{\partial^2}{\partial x^2} - \varepsilon A(x, \varepsilon) \dfrac{\partial}{\partial x} - \varepsilon\ \frac{\partial}{\partial t}$ on the ansatz (9) we get

$$DU(x, t, \varepsilon) = D\overline{u}(x, X^*(t, \varepsilon), \varepsilon) + \left[\frac{\partial^2}{\partial \xi^2} + \left(\frac{\partial X^*}{\partial t}(t, \varepsilon) - \varepsilon A(x, \varepsilon)\right) \frac{\partial}{\partial \xi} - \varepsilon\ \frac{\partial}{\partial t}\right] Q(\xi, t, \varepsilon).$$

Equating the sum of coefficients of the equal powers of ε to zero, we easily obtain equations for determining all terms of the asymptotic.

It is obvious that the equation for the zeroth order regular function $\overline{u}_0(x, y)$ coincides with the reduced equation (3). Thus, let us put

$$\overline{u}_0^{(\pm)}(x, y) = \varphi^{(\pm)}(x, y).$$

Extracting leading terms from expansions (10) and (12), we obtain the problem for determination of the boundary functions $Q_0^{(\pm)}$ $(v_0(t) = x_0'(t))$, namely

$$\frac{\partial^2 Q_0^{(\pm)}}{\partial \xi^2} + (v_0(t) - A(x_0(t),0))\frac{\partial Q_0^{(\pm)}}{\partial \xi} =$$
$$= \int\limits_a^b [g(\overline{u}_0^{(\pm)}(x_0(t),x_0(t)) + Q_0^{(\pm)}(\xi,t), \overline{u}_0^{(\pm)}(s,x_0(t)), x_0(t), s, 0) -$$
$$-g(\overline{u}_0^{(\pm)}(x_0(t),x_0(t)), \overline{u}_0^{(\pm)}(s,x_0(t)), x_0(t), s, 0)]\, ds, \quad \xi \in \mathbb{R}^{\pm}, \tag{14}$$

$$Q_0^{(\pm)}(0,t) + \overline{u}_0^{(\pm)}(x_0(t),x_0(t)) = \varphi_0(x_0(t)), \tag{15}$$

$$Q_0^{(-)}(-\infty,t) = Q_0^{(+)}(+\infty,t) = 0. \tag{16}$$

It is clear that the solutions of problems (14)-(16) do not explicitly depend on the argument t and depend on x_0 as parameter. Therefore according to **Remark 1** we can use the notation

$$Q_0^{(\pm)}(\xi,t) \equiv Q_0^{(\pm)}(\xi,x_0(t)).$$

In order to use known results for problems (14)-(16) we introduce the continuous function

$$\tilde{u}(\xi,x_0) \equiv \begin{cases} \varphi^{(-)}(x_0,x_0) + Q_0^{(-)}(\xi,x_0), & \xi < 0, \\ \varphi_0(x_0), \quad \xi = 0, \\ \varphi^{(+)}(x_0,x_0) + Q_0^{(+)}(\xi,x_0), & \xi > 0. \end{cases}$$

Now we can rewrite problems (14) - (16) into the form

$$\begin{gathered} \frac{\partial^2 \tilde{u}}{\partial \xi^2} + (v_0 - A(x_0,0))\frac{\partial \tilde{u}}{\partial \xi} = \int_b^a g(\tilde{u}, \varphi(s,x_0),\ x_0, s, 0)\, ds, \quad \xi \in \mathbb{R}, \\ \tilde{u}(0,x_0) = \varphi_0(x_0), \quad \tilde{u}(-\infty,x_0) = \varphi^{(-)}(x_0,x_0), \quad \tilde{u}(+\infty,x_0) = \varphi^{(+)}(x_0,x_0). \end{gathered} \tag{17}$$

The equation in (17) is a second order ODE and from **Condition I** it follows that $\varphi^{(\pm)}(x_0,x_0)$ are the sadle points. This problem is well studied (see, for example, [7], [9]), and we have the following result.

Lemma 1 *For all $x_0 \in (a,b)$ there exists a unique v_0 such that problem (17) has a unique solution $\tilde{u}$ satisfying the estimates*

$$\left|\tilde{u}(\xi,x_0) - \varphi^{(\pm)}(x_0,x_0)\right| \le Ce^{-\nu|\xi|}, \tag{18}$$

where C and ν are some positive constants. The function $v_0(x_0)$ satisfies the relation

$$v_0(x_0) = \frac{\int\limits_{\varphi^{(-)}(x_0,x_0)}^{\varphi^{(+)}(x_0,x_0)} \left[\int\limits_a^b g(u,\varphi(s,x_0),x_0,s,0)\, ds\right] du}{\int\limits_{-\infty}^{+\infty} \left(\frac{\partial \tilde{u}}{\partial \xi}(\xi,x_0)\right)^2 d\xi} + A(x_0,0). \tag{19}$$

From **Lemma 1** it follows that $Q_0^{(\pm)}(\xi,t)$ have the exponential estimate

$$\left|Q_0^{(\pm)}(\xi,t)\right| \le Ce^{-\nu|\xi|}, \tag{20}$$

and the location of front can be determined as a solution of the initial value problem

$$
\begin{aligned}
&x_0' = v_0(x_0),\\
&x_0(0) = x_{00},
\end{aligned}
\tag{21}
$$

where x_{00} determines the initial location of the front. The roots of $v_0(x_0)$ determine the location of stationary solution with internal layers (see [5]). At the present work we consider just motion of fronts (without approaching to stationary solutions), therefore we assume

Condition II. *Let* $v_0(x_0) > 0$ *for any* $x_0 \in [a, b]$.

Problem (21) has a solution for $t \in [0, T]$, where T is defined by the estimate $x_0(T) < b$ (the front moves to the right boundary).

¿From the solvability of problem (17) it follows the zeroth order C^1-matching condition (see (11))

$$
\frac{\partial Q_0^{(-)}}{\partial \xi}(0, x_0(t)) = \frac{\partial Q_0^{(+)}}{\partial \xi}(0, x_0(t)) \tag{22}
$$

is satisfied.

We now turn to the analysis of the linear problems determining the higher order terms in the asymptotic expansion.

The first order regular expansion term $\overline{u}_1(x, y, t)$ of (9) has to be determined for any smooth function $y(t) \in (a, b)$ for $t \in [0, T]$ therefore we denote it by $\overline{u}_{1y}(x, t)$ and determined by the operator equation

$$
\begin{gathered}
P(x, y)\overline{u}_{1y}(x, t) + + \int\limits_a^b g_v(\overline{u}_0(x, y), \overline{u}_0(s, y), x, s, 0)\overline{u}_{1y}(s, y)\, ds +\\
D_1(x, y) + T_1(x, y) = \frac{\partial \overline{u}_0}{\partial t}(x, y) = \frac{\partial \overline{u}_0}{\partial y}(x, y)\frac{\partial y}{\partial t}
\end{gathered}
\tag{23}
$$

If $y(t)$ is determined, then $\overline{u}_{1y}(x, t)$ is a known function of x, t and we use for it the notation For a shortness we use for this term the notation $\overline{u}_1(x, y, t)$ where y denote the point where $\overline{u}_1(x, y, t)$ is discontinues.

By analogy with **Condition I** one can rewrite problem (23) in the form of two coupled linear integro-differential equations

$$
\begin{aligned}
&\overline{u}_{1y}^{(-)} + \int\limits_a^y K^{(--)}(x, s, y)\overline{u}_{1y}^{(-)}(s, t)\, ds +\\
&\qquad + \int\limits_y^b K^{(-+)}(x, s, y)\overline{u}_{1y}^{(+)}(s, t)\, ds = f_{1y}^{(-)}(x, t), \quad a < x < y,\\
&\overline{u}_{1y}^{(+)} + \int\limits_a^y K^{(+-)}(x, s, y)\overline{u}_{1y}^{(-)}(s, t)\, ds +\\
&\qquad\qquad + \int\limits_y^b K^{(++)}(x, s, y)\overline{u}_1^{(+)}(s, t)\, ds = f_{1y}^{(+)}(x, t), \quad y < x < b.
\end{aligned}
\tag{24}
$$

Here for $i, j \in \{-,+\}$ we denoted

$$K^{(ij)}(x,s,y) \;=\; \frac{g_v(\overline{u}_0^{(i)}(x,y), \overline{u}_0^{(j)}(s,y), x, s, 0)}{P^{(i)}(x,y)},$$

$$f_{1y}^{(i)}(x,t) \;=\; \frac{\dfrac{\partial \overline{u}_0}{\partial y}(x,y)\dfrac{\partial y}{\partial t} - D_1^{(i)}(x,y) - T_1^{(i)}(x,y)}{P^{(i)}(x,y)} \qquad (y = y(t)).$$

Suppose that the following condition is satisfied.

Condition III. *Let the system of the coupled integral inequalities*

$$w^{(-)}(x,y) \;+\; \int\limits_a^y K^{(--)}(x,s,y)w^{(-)}(s)\,ds + \int\limits_y^b K^{(-+)}(x,s,y)w^{(+)}(s)\,ds > 0, \quad a \le x \le y,$$

$$w^{(+)}(x,y) \;+\; \int\limits_a^y K^{(+-)}(x,s,y)v^{(-)}(s)\,ds + \int\limits_y^b K^{(++)}(x,s,y)v^{(+)}(s)\,ds > 0, \quad y \le x \le b,$$

for all $y \in (a,b)$ has a positive solution .

Remark I. *Condition III is fulfilled if all eigenvalues of the following problem*

$$\lambda w^{(-)}(x,y) + \int\limits_a^y K^{(--)}(x,s,y)w^{(-)}(s)\,ds + \int\limits_y^b K^{(-+)}(x,s,y)w^{(+)}(s)\,ds = 0, \quad a \le x \le y,$$

$$\lambda w^{(+,y)}(x) + \int\limits_a^y K^{(+-)}(x,s,y)v^{(-)}(s)\,ds + \int\limits_y^b K^{(++)}(x,s,y)v^{(+)}(s)\,ds = 0, \qquad y \le x \le b,$$

have estimate $|\lambda| < 1$.

We can also use the sufficient condition

$$P(x,y) + \int\limits_a^b g_v(\varphi(x,y), \varphi(s,y), x, s, 0)\,ds > 0 \quad \forall \quad (x,y) \in [a,b],$$

that can be easily checked.

If **Condition III** is fulfilled then for all y problem (24) has unique solution $\overline{u}_1^{(\pm)}(x,y)$ with discontinuity at the point y (see [10]).

The internal layer functions $Q_1^{(\pm)}(\xi,t)$ are the solutions of the equations

$$\begin{gathered}\frac{\partial^2 Q_1^{(\pm)}}{\partial \xi^2} + \tilde{v}_0(x_0)\frac{\partial Q_1^{(\pm)}}{\partial \xi} - \\ -Q_1^{(\pm)} \int\limits_a^b g_u(\tilde{u}(\xi,t), \varphi(s,x_0), x_0, s, 0)\,ds = q_1^{(\pm)}(\xi,t), \quad \xi \in \mathbb{R}^{\pm},\end{gathered} \tag{25}$$

where

$$q_1^{(\pm)}(\xi,t) = -x_1'\frac{\partial \tilde{u}}{\partial \xi}(\xi,x_0) - (\frac{\partial A}{\partial x}(x_0,0)(x_1+\xi) + \frac{\partial A}{\partial \varepsilon}(x_0,0))\tilde{v}_0(x_0) + S_1^{(\pm)}(\xi,t) + \Theta_0^{(\pm)}(\xi,t),$$

and

$$\tilde{v}_0(t) \equiv v_0(t) - A(x_0(t), 0).$$

The equations (25) is considered with the boundary conditions

$$\begin{aligned} Q_1^{(-)}(0,t) &= \overline{u}_1^{(-)}(x_0, x_0) + x_1 \left[\frac{\partial \overline{u}_0^{(-)}}{\partial x}(x_0, x_0) + \frac{\partial \overline{u}_0^{(-)}}{\partial y}(x_0, x_0) \right] \equiv p_1^{(-)}(t), \\ Q_1^{(+)}(0,t) &= \overline{u}_1^{(+)}(x_0, x_0) + x_1 \left[\frac{\partial \overline{u}_0^{(+)}}{\partial x}(x_0, x_0) + \frac{\partial \overline{u}_0^{(+)}}{\partial y}(x_0, x_0) \right] \equiv p_1^{(+)}(t), \end{aligned} \tag{26}$$

$$Q_1^{(-)}(-\infty, t) = Q_1^{(+)}(+\infty, t) = 0. \tag{27}$$

Using that $\frac{\partial \tilde{u}}{\partial \xi}(\xi, t)$ is the solution of the homogeneous equation for (25) we can obtain the following integral representation of solution of problem (25)–(26):

$$Q_1^{(\pm)}(\xi, t) = z(\xi, x_0) \left\{ p_1^{(\pm)}(t) - \int_0^{\xi} \frac{e^{-\tilde{v}_0(x_0)\eta}}{z^2(\eta, x_0)} \left[\int_{\eta}^{\pm\infty} z(\chi, x_0) e^{\tilde{v}_0(x_0)\chi} q_1^{\pm}(\chi, t) d\chi \right] d\eta \right\}, \tag{28}$$

where

$$z(\xi, x_0) \equiv \frac{\partial \tilde{u}}{\partial \xi}(\xi, x_0) \left(\frac{\partial \tilde{u}}{\partial \xi}(0, x_0) \right)^{-1}.$$

¿From the definition of $q_1(\xi, t)$ and the exponential estimate (20) for $Q_0(\xi, t)$ it follows that $|q_1(\xi, t)| \leq Ce^{-\nu|\xi|}$ for all $\xi \in \mathbb{R}^{\pm}$. Now it is obvious that, if the function $x_1(t)$ is known, the linear problems (25)-(26) have unique solutions and these solutions also have the exponential estimates at infinity

$$\left| Q_1^{(\pm)}(\xi, t) \right| \leq Ce^{-\nu|\xi|}.$$

To find function $x_1(t)$ we shall use the first order C^1-matching condition (see (11))

$$\frac{\partial Q_1^{(-)}}{\partial \xi}(0, t) + \frac{\partial \overline{u}_0^{(-)}}{\partial x}(x_0(t), x_0(t)) = \frac{\partial Q_1^{(+)}}{\partial \xi}(0, t) + \frac{\partial \overline{u}_0^{(+)}}{\partial x}(x_0(t), x_0(t)). \tag{29}$$

¿From the integral representation (28) and condition (29) we obtain the following equation for function $x_1(t)$:

$$x_1' - B(x_0(t))x_1 = F_1(x_0(t)), \tag{30}$$

and

$$B(x_0) = \frac{\int\limits_{-\infty}^{+\infty} \tilde{u}_\xi(\xi, x_0) e^{\tilde{v}_0(x_0)\xi} N(\xi, x_0)\, d\xi - \tilde{u}_\xi(0, x_0)\tilde{v}_0(x_0) D(x_0)}{\int\limits_{-\infty}^{+\infty} \tilde{u}_\xi^2(\xi, x_0) e^{\tilde{v}_0(x_0)\xi}\, d\xi}$$

where
$$N(\xi,x_0)=-A_x(x_0,0)\tilde{v}_0(x_0)+g(\tilde{u}(\xi,x_0),\varphi^{(-)}(x_0,x_0),x_0,s,0)-$$
$$-g(\tilde{u}(\xi,t),\varphi^{(+)}(x_0,x_0),x_0,s,0)+\varphi_x^{(-)}(x_0,x_0)\int\limits_a^{x_0}g_u(*)ds+$$
$$+\varphi_x^{(+)}(x_0,x_0)\int\limits_{x_0}^{b}g_u(*)ds+\int\limits_a^{x_0}g_v(*)\varphi_y^{(-)}(s,x_0)ds+$$
$$+\int\limits_{x_0}^{b}g_v(*)\varphi_y^{(+)}(s,x_0)ds+\int\limits_a^{b}g_x(*)ds$$

$((*)\equiv(\tilde{u}(\xi,x_0),\varphi(s,x_0),x_0,s,0))$,
$$D(x_0)=\varphi_x^{(+)}(x_0,x_0)-\varphi_x^{(-)}(x_0,x_0)+\varphi_y^{(+)}(x_0,x_0)-\varphi_y^{(-)}(x_0,x_0)$$

and

$$F_1(x_0)=\int\limits_{-\infty}^{+\infty}\tilde{u}_\xi(\xi,x_0)e^{\tilde{v}_0(x_0)\xi}\{-(A_x(x_0,0)\xi+A_\varepsilon(x_0,0))\tilde{v}_0(x_0)+\xi\{\varphi_x^{(-)}(x_0,x_0)\int\limits_a^{x_0}g_u(*)ds$$
$$+\varphi_x^{(+)}(x_0,x_0)\int\limits_{x_0}^{b}g_u(*)ds\}+\bar{u}_1^{(-)}(x_0,x_0)\int\limits_a^{x_0}g_u(*)ds+\bar{u}_1^{(+)}(x_0,x_0)\int\limits_{x_0}^{b}g_u(*)ds+$$
$$+\int\limits_a^{b}g_v(*)\bar{u}_1(s,x_0)ds+\xi\int\limits_a^{b}g_x(*)ds+\int\limits_a^{b}g_\varepsilon(*)ds+\Theta_0(\xi,x_0)\}\,d\xi+$$
$$+\tilde{u}_\xi(0,x_0)\tilde{v}_0(x_0)(\overline{u}_1^{(+)}(x_0,x_0)-\overline{u}_1^{(-)}(x_0,x_0))$$

($\Theta_0(\xi,x_0)$ is the integral term in the left-hand side of (13)).

For equation (30) we have initial condition
$$x_1(0)=0.$$

The problems to determine the terms $\overline{u}_k(x,y)$ and $Q_k(\xi,t)$ for $k\geq 2$ have the same structure as problem (23) and (25)-(26), respectively. Thus, we get that all these problems are always solvable. Particularly, the functions $Q_k(\xi,t)$ can be represented in the form (28) where index 1 has to be replaced by k, and $q_k^{(\pm)}(\xi,t)$ are known on each step functions with the exponential estimate. Thus, we have
$$\left|Q_k^{(\pm)}(\xi,t)\right|\leq Ce^{-\nu|\xi|},$$
where $C>0$ and $\nu>0$ are certain constants independent of ξ.

Using the k-th order C^1-matching condition (see (11)) we get the equation for $x_k(t)$ which is similar to equation (30) for $x_1(t)$
$$x_k'-B(x_0(t))x_k=F_k(t),$$
where $F_k(t)$ is known on the each step function, with additional condition $x_k(0)=0$.

If input dates are sufficiently smooth then we can continue our construction to any order of n, and our formal asymptotics satisfies the problem to order of ε^{n+1}.

3. Existence result

To validate the asymptotics constructed above, we invoke the asymptotic method of differential inequalities [11], which was initially proposed for PDE's, and got some extension for integro-differential equations(see [5],[8]and [10]. This approach is based on well known differential inequalities technique and we recall the definition of upper and lower solutions to problem (1), (2).

Definition 1. *A function $\beta(x,t,\varepsilon) \in C\{[a,b]\times[0,T]\} \cap C^{2,1}\{(a,x^\star(t)]\times(0,T]\} \cap C^{2,1}\{[x^\star(t),b)\times(0,T]\}$, where $x^\star(t)\in(a,b)$ for $t\in[0,T]$ is a smooth function, is called an upper solution to problem (1), (2) if*

1) $L[\beta]\le 0$ for all $(x,t)\in\{(a,x^\star]\times(0,T]\}\cap\{[x^\star,b)\times(0,T]\}$,

2) $\dfrac{\partial\beta}{\partial x}(x^\star+0,t,\varepsilon)-\dfrac{\partial\beta}{\partial x}(x^\star-0,t,\varepsilon)\le 0$,

3) $\dfrac{\partial\beta}{\partial x}(a,t,\varepsilon)\le 0$ and $\dfrac{\partial\beta}{\partial x}(b,t,\varepsilon)\ge 0$,

4) $\beta(x,0,\varepsilon)\ge u^0(x,\varepsilon)$,

where L is the integro-differential operator of equation (1). Similarly, the function $\alpha(x,t,\varepsilon)$ belonging to the same class of smoothness is called a lower solution if it satisfies to the conversed inequalities.

The proof of the existence of a solution to problem (1), (2) relies on the following theorem of differential inequalities, which is a slight extension of the corresponding theorem in [2] (see Chapter 2, Sect.2.7) (see also [12], [13] and [14]).

Theorem 1. *Assume that there exist the functions $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ such that the following conditions are valid:*

(a) $\alpha(x,t,\varepsilon)$ and $\beta(x,t,\varepsilon)$ are the lower and upper solutions to problem (1), (2), respectively;

(b) $\alpha(x,t,\varepsilon)\le\beta(x,t\varepsilon)$ for all $(x,t)\in\{[a,b]\times[0,T]\}$;

(c) $A(x,\cdot)\in C^1([a,b]$, $g(u,v,x,s,\cdot)$ and $g_u(\dots)$, $g_v(\dots)\in C([\alpha(x,t,\cdot),\beta(x,t,\cdot)]\times[\alpha(s,t,\cdot),\beta(s,t,\cdot)]\times[a,b]^2)$;

(d) $g_v(\dots)\le 0$ for all $(u,v,x,s)\in[\alpha(x,t,\cdot),\beta(x,t,\cdot)]\times[\alpha(s,t,\cdot),\beta(s,t,\cdot)]\times[a,b]^2$.

Then problem (1), (2) has unique classical solution $u(x,t,\varepsilon)$ such that $\alpha(x,t,\varepsilon)\le u(x,t,\varepsilon)\le\beta(x,t,\varepsilon)$ for $(x,t)\in\{[a,b]\times(0,T]\}$.

We note once again that Theorem 1 can be used in our case only if the function g in Eq. (1) satisfies to so called quasimonotonicity assumption(d). Therefore we assume the following condition.

Condition IV. *The function $g(u,v,x,s,\varepsilon)$ is monotonically nonincreasing with respect to v for all admissible values of its arguments.*

The lower and upper solutions to problem (1), (2) are sought in the form

$$\begin{aligned} \alpha_n(x,t,\varepsilon) &= U_{(n+1)\alpha}(x,t,\varepsilon)+\varepsilon^{n+1}\left[-q(x,y)+\tilde{Q}_{(n+1)\alpha}(\xi_\alpha,t)\right] \\ &\quad -\varepsilon^{n+1}(e^{-\kappa\xi^{(-)}}+e^{-\kappa\xi^{(+)}}), \\ \beta_n(x,t,\varepsilon) &= U_{(n+1)\beta}(x,t,\varepsilon)+\varepsilon^{n+1}\left[q(x,y)+\tilde{Q}_{(n+1)\beta}(\xi_\beta,t)\right] \\ &\quad +\varepsilon^{n+1}(e^{-\kappa\xi^{(-)}}+e^{-\kappa\xi^{(+)}}), \end{aligned} \tag{31}$$

where $U_{(n+1)\alpha}(x,\varepsilon)$ and $U_{(n+1)\beta}(x,\varepsilon)$ are $(n+1)$-st order $(n\ge 0)$ partial sums of series (9) with $y = x^\star_\alpha(t,\varepsilon)$ and $y = x^\star_\beta(t,\varepsilon)$, respectively, where

$$x^\star_\alpha(t,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k x_k(t) + \varepsilon^{n+1} x_{(n+1)\alpha}(t), \quad x^\star_\beta(t,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k x_k(t) + \varepsilon^{n+1} x_{(n+1)\beta}(t),$$

and $x_{(n+1)\alpha}$ and $x_{(n+1)\beta}$ to be determined later. The stretched variable ξ we replaced here by $\xi_\alpha = [x - x^\star_\alpha(t,\varepsilon)]/\varepsilon$ in α_n, and by $\xi_\beta = [x - x^\star_\beta(t,\varepsilon)]/\varepsilon$ in β_n.

We recall that function $x_0(t)$ is such that $a + \Delta \le x_0(t) \le b - \Delta$ ($\Delta > 0$ is some positive number) for all $t \in [0, T]$.

The function $q(x, y) \ge q_0 > 0$ for all $(x, y) \in [a, b] \times [x_0(t) - \Delta, x_0(t) + \Delta]$ is positive solution to integral equation

$$P(x,y)q(x,y) + \int_a^b g_v(\overline{u}_0(x,y), \overline{u}_0(s,y), x, s, 0) q(s,y) ds = 1 \tag{32}$$

The existence of such positive solution of equation (32) follows from **Condition III**.

The function $x_{(n+1)\beta}$ is defined from the problem

$$x'_{(n+1)\beta} - B(x_0(t)) x_{(n+1)\beta} = F_{(n+1)\beta}(t) - \sigma, \quad x_{(n+1)\beta}(0) = -\delta,, \tag{33}$$

where

$$F_{(n+1)\beta}(t) = q^{(-)}(x_0,x_0) \int_a^{x_0} g_u(*) ds + q^{(+)}(x_0,x_0) \int_{x_0}^{b} g_u(*) ds +$$
$$+ \int_a^b g_v(*) q(s,x_0) ds + \tilde{u}_\xi(0,x_0) \tilde{v}_0(x_0)(q^{(+)}(x_0,x_0) - q^{(-)}(x_0,x_0))$$

($(*) \equiv (\tilde{u}(\xi, x_0), \varphi(s, x_0), x_0, s, 0)$), and σ and δ are some positive numbers. If we choose δ sufficiently large we have

$$x_{(n+1)\beta}(t) < 0, \quad t \in [0, T] \tag{34}$$

We determine the function $\tilde{Q}_{(n+1)\beta}(\xi, t)$ from the problem which is similar to the problems (25)-(27)

$$\begin{aligned} \frac{d^2 \tilde{Q}^{(\pm)}_{(n+1)\beta}}{d\xi^2} + \tilde{v}_0(t) \frac{d\tilde{Q}^{(\pm)}_{(n+1)\beta}}{d\xi} - \tilde{Q}^{(\pm)}_{(n+1)\beta} \int_a^b g_u(\tilde{u}(\xi,t), \varphi(s,x_0), x_0, s, 0)\, ds = \\ = q^{(\pm)}_{(n+1)\beta}(\xi, t), \quad \xi \in \mathbb{R}^\pm, \end{aligned} \tag{35}$$

$$\begin{aligned} \tilde{Q}^{(-)}_{(n+1)\beta}(0,t) = q^{(-)}(x_0,x_0) + x_{(n+1)\beta} \left[\frac{\partial \overline{u}_0^{(-)}}{\partial x}(x_0,x_0) + \frac{\partial \overline{u}_0^{(-)}}{\partial y}(x_0,x_0) \right], \\ \tilde{Q}^{(+)}_{(n+1)\beta}(0,t) = q^{(+)}(x_0,x_0) + x_{(n+1)\beta} \left[\frac{\partial \overline{u}_0^{(+)}}{\partial x}(x_0,x_0) + \frac{\partial \overline{u}_0^{(+)}}{\partial y}(x_0,x_0) \right], \end{aligned} \tag{36}$$

$$\tilde{Q}^{(-)}_{(n+1)\beta}(-\infty, t) == \tilde{Q}^{(+)}_{(n+1)\beta}(+\infty, t) = 0, \tag{37}$$

where

$$q^{(\pm)}_{(n+1)\beta}(\xi,t) = -x'_{(n+1)\beta} \frac{\partial \tilde{u}}{\partial \xi}(\xi, x_0) - \frac{\partial A}{\partial x}(x_0, 0) x_{(n+1)\beta} \tilde{v}_0(x_0).$$

The solution of problem (35)–(37) can be obtained explicitly.

Now we can check that the function $\beta_n(x,t,\varepsilon)$ given by the representation (31) satisfies **Definition 1**.

From the structure of $\beta_n(x,t,\varepsilon)$ it follows that it satisfies the differential inequality. We have

$$L[\beta_n] \le -\varepsilon^{n+1} + o(\varepsilon^{n+1}).$$

Thus, for sufficiently small values of the parameter ε we obtain $L[\beta_n] < 0$.

From representation (31) for $\beta_n(x,t,\varepsilon)$ using the equation for $x_{(n+1)\beta}$ (see (33) we get that in the point $x^\star_\beta(t,\varepsilon)$ the condition of the derivative's jump is satisfied

$$\frac{d\beta}{dx}(x^\star_\beta(t,\varepsilon)+0,t,\varepsilon) - \frac{d\beta}{dx}(x^\star_\beta(t,\varepsilon)-0,t,\varepsilon) =$$

$$= \varepsilon^n(x'_{(n+1)\beta} - B(x_0(t))x_{(n+1)\beta} - F_{(n+1)\beta}(t))\tilde{u}_\xi^{-1}(0,x_0)\int_{-\infty}^{+\infty}\tilde{u}_\xi^2(\xi,x_0)e^{\tilde{v}_0(x_0)\xi}\,d\xi + O(\varepsilon^{n+1})$$

$$= -\varepsilon^n\sigma\tilde{u}_\xi^{-1}(0,x_0)\int_{-\infty}^{+\infty}\tilde{u}_\xi^2(\xi,x_0)e^{\tilde{v}_0(x_0)\xi}\,d\xi + O(\varepsilon^{n+1}) < 0$$

for sufficiently small $\varepsilon > 0$.

Therefore the function $\beta_n(x,\varepsilon)$ satisfies the definition of the upper solution and it is an upper solution for problem (1), (2).

Similarly it can be shown that function $\alpha_n(x,\varepsilon)$ is lower solution for problem (1), (2).

The proof that $\alpha_n(x,\varepsilon)$ and $\beta_n(x,\varepsilon)$ are ordered we show that difference $\beta_n(x,\varepsilon) - \alpha_n(x,\varepsilon) > 0$ by the same way as in [10], [11].

Thus, $\alpha_n(x,t,\varepsilon)$ and $\beta_n(x,t,\varepsilon)$ satisfy all conditions of Theorem 1. Therefore, problem (1), (2) has a solution $u(x,t,\varepsilon)$ such that $\alpha_n(x,t,\varepsilon) \le u(x,t,\varepsilon) \le \beta_n(x,t,\varepsilon)$. Taking into account that $\beta_n(x,t,\varepsilon) - \alpha_n(x,t,\varepsilon) = O(\varepsilon^n)$ we find that $u(x,t,\varepsilon) = \alpha_n(x,t,\varepsilon) + O(\varepsilon^n)$. In the case when $n \ge 1$, we can neglect inessential terms and write $u(x,t,\varepsilon) = U_{n-1}(x,t,\varepsilon) + O(\varepsilon^n)$ with $x^\star(t,\varepsilon) = \sum_{k=0}^{n+1}\varepsilon^k x_k(t)$. Thus, we have proved the following theorem.

Theorem 2. *Assume that $A \in C^{n+2}([a,b])$, $g \in C^{n+2}(\mathbb{R}^2 \times [a,b]^2)$ $(n \ge 0)$ and for sufficiently small ε, and Conditions I-IV are fulfilled. Then for sufficiently small values of the parameter $\varepsilon > 0$ problem (1), (2) with initial function*

$$\alpha_1(x,0,\varepsilon) \le u^0(x,\varepsilon) \le \beta_1(x,0,\varepsilon)$$

has a classical solution $u = u(x,t,\varepsilon)$ for $t \in [0,T]$ such that

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = \begin{cases} \varphi^{(-)}(x,x_0(t)) & \text{for} \quad a \le x < x_0(t) < b \\ \varphi^{(+)}(x,x_0(t)) & \text{for} \quad x_0(t) < x \le b. \end{cases}$$

Moreover, if smoothness condition on coefficients A and g are fulfilled for $n \ge 2$ and

$$\alpha_{n-1}(x,0,\varepsilon) \le u^0(x,\varepsilon) \le \beta_{n-1}(x,0,\varepsilon)$$

then $u(x,t,\varepsilon) = U_{n-1}(x,t,\varepsilon) + O(\varepsilon^n)$ *where*

$$U_{n-1}(x,t,\varepsilon) = \begin{cases} \sum\limits_{k=0}^{n-1} \varepsilon^k \left[\overline{u}_k^{(-)}(x, x_n^\star(t,\varepsilon)) + Q_k^{(-)}(\xi,t)\right] + \sum\limits_{k=1}^{n-1} \varepsilon^k \Pi_k^{(-)}(\xi^{(-)}), \ a \le x \le x_n^\star(t,\varepsilon), \\ \sum\limits_{k=0}^{n-1} \varepsilon^k \left[\overline{u}_k^{(+)}(x, x_n^\star(t,\varepsilon)) + Q_k^{(+)}(\xi,t)\right] + \sum\limits_{k=1}^{n-1} \varepsilon^k \Pi_k^{(+)}(\xi^{(+)}), \ x_n^\star(t,\varepsilon) < x \le b \end{cases}$$

with $\xi = [x - x_n^\star(t,\varepsilon)]/\varepsilon$ *and* $x_n^\star(t,\varepsilon) = \sum\limits_{k=0}^{n+1} \varepsilon^k x_k(t)$.

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